QCD. Chiral symmetry breaking

- Donoghue, Golowich, Holstein
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- A. Zee (Group Th. in a nutshell) Part V

The last missing part of the SM is Quantum ChromoDynamics (QCD).

QCD is a gauge theory, with gauge group $SU(3)$, with the charges refered to as "color". Quarks are color triplets, while leptons are singlets. So under the SM group

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

the matter content is

$$q_L \in (3, 2, 1/6) \qquad u_R \in (3, 1, 2/3) \qquad d_R \in (3, 1, -1/3)$$

$$l_L \in (1, 2, -1/2) \qquad e_R \in (1, 1, -1)$$

$$H \in (1, 2, 1/2)$$

QCD is a vector-like gauge theory, with L & R quarks in the same (3) representation.

· Since SU(3) has 8 generators, QCD contains 8 real spin-1 force carriers, the gluons.

The Lagrangian is obtained from our discussion on non-abelian gauge groups.

Using the fund. rep

$$T^a = \frac{1}{2} \lambda^a$$

with λ^a being 8 3×3 matrices (Gell-Mann matrices), defining

$$G_\mu \equiv G_\mu^a T^a$$

and the gluon field strength

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i [G_\mu, G_\nu]$$

we have

$$\mathcal{L}_{QCD}^{gluons} = - \frac{1}{2g_s^2} tr[G_{\mu\nu} G^{\mu\nu}] + \theta_{CP} \, tr[G_{\mu\nu} \tilde{G}^{\mu\nu}]$$

with $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}$.

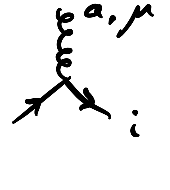
- g_s is a new parameter of the QCD Lagrangian, the 'strong coupling constant'.

- The second term breaks CP. It is in principle allowed, but experimentally $\theta_{CP} < 10^{-10}$. So for the rest of the lectures we will ignore it. Since we do not understand why it is so small, this is a puzzle of the SM refered to "strong CP problem".

- The quark-gluon interaction is the one dictated by gauging.

$$D_\mu q = \partial_\mu q - i g_s G^a_\mu T^a q$$

with Feynman rule



$$= i g_s \gamma^\mu (T^a)_i{}^j$$

with 'i, j' color indices in the triplet and 'a' an adjoint index.

- The running of α_s.

The QCD sector is <u>very</u> different from the QED and EW sectors. The reason is that QCD runs into strong coupling in the IR.

We have seen how coupling constants are not constant. We can use the renormalization group to resum large logarithms into a redefinition of the coupling, according to

$$\mu \frac{de}{d\mu} = \beta(e) \, ,$$

where e is the running coupling & μ the typical scale of a process.
$\beta(e)$ is the β-function, and determines the evolution.

In QED, we've seen that

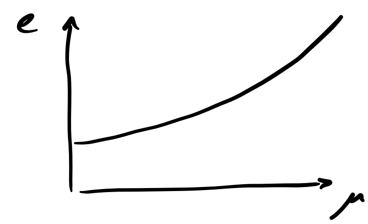
$$\beta(e) = \frac{e^3}{12\pi^2}$$

So

$$\frac{1}{e^3}\,de = \frac{1}{12\pi^2}\frac{d\mu}{\mu} \quad\rightarrow\quad \frac{1}{e^2(\mu)} = \frac{1}{e^2(\mu_0)} - \frac{1}{12\pi^2}\log\frac{\mu}{\mu_0}$$

$$\rightarrow\quad e^2(\mu) = \frac{e^2}{1-\frac{e^2}{12\pi^2}\log\left(\frac{\mu}{\mu_0}\right)}$$

The coupling increases at high energies



and it has a "Landau pole" at <u>very</u> high energies.

- The calculation of the QCD beta function is a bit more involved, and you will see it / you have seen it in QFT III.

We shall only state the result:

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3} - \frac{2}{3}n_f\right)$$

where n_f is the number of Dirac fermions.

In this calculation we've assumed $N_c = 3$, so $SU(3)$, and that fermions are in the fundamental.

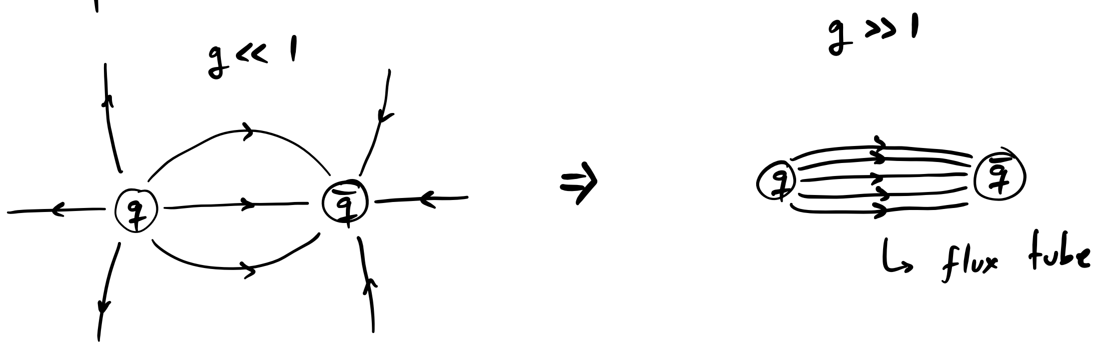
Notice that if $n_f < 17$, then $\beta(g) < 0$.

The fact that the beta function is negative means that QCD becomes non-pert. at low energies.

Taking $\alpha_s = \frac{g_s^2}{4\pi} \simeq 0.12$ at $\mu_0 \simeq m_Z$,

then $g_s \simeq 4\pi$ at $\mu \equiv \Lambda_{QCD} \sim 100$ MeV.

So at low energies the theory becomes strongly coupled.



$V(r) \sim -\frac{e^2}{r}$ $\qquad$ $V(r) \sim \sigma r$

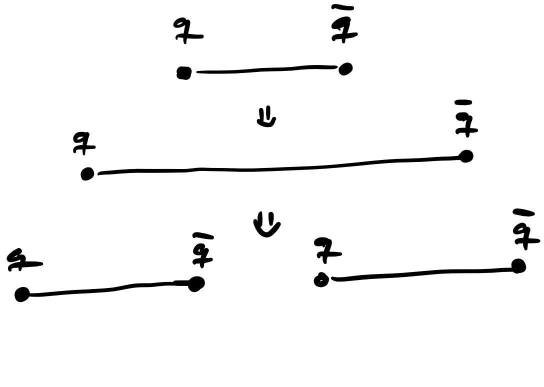
It is assumed (not proven!) that Yang-Mills theories, and certainly QCD, generate a mass gap.

The dynamically generated scale Λ_{QCD} changes the potential between two charges from $1/r$ to a confining potential $\sim r$.

$$V(r) \simeq \underbrace{\sigma}_{\sim \Lambda_{QCD}^2} r$$

with σ being the "string tension".

Given that energy grows with the quark separation, eventually there is enough energy stored to create a $\bar{q}q$ pair to break the string.



so the quarks are confined.

Thus, asymptotic states in QCD are not quarks and gluons, but bound states we call hadrons.

- Light mesons

The mesons are pseudoscalar states with the quantum numbers of a $q\bar{q}$ pair.

There are 8 of them that look lighter than the rest of resonances:

$\omega^0 \sim 1020$ MeV

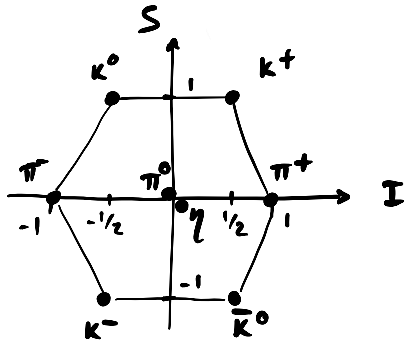
$p, n \sim 940$ MeV

$\rho, \omega^0 \sim 770$ MeV

$$\left.\begin{array}{ll} \eta & \sim 548 \text{ MeV} \\ K^\pm, K^0, \bar{K}^0 & \sim 498 \text{ MeV} \\ \pi^\pm, \pi^0 & \sim 140 \text{ MeV} \end{array}\right\} \text{light mesons}$$

Moreover, studying scattering, decays, etc, it seems that strong interactions preserve two quantum numbers: isospin (I) & strangeness (s)

If we plot these resonances in terms of these quantum numbers, one gets



which certainly looks as if the mesons were organized in the adjoint of SU(3)! With isospin & strangeness being the two diagonal generators.

This seems odd, while quantum numbers fall in a SU(3) rep, the masses of the mesons are different, so if there is a symmetry, it cannot be exact.

To understand what this symm might be, let's look at the global symm of QCD Lagrangian in terms of quarks & gluons.

- Chiral symmetry

The QCD Lagrangian with one flavor, $n_f = 1$, is

$$\mathcal{L}^{QCD}_{n_f=1} = -\frac{1}{2}\,tr(G_{\mu\nu}G^{\mu\nu}) + \bar{q}\,i\not{D}\,q - m_q\,\bar{q}q$$

This theory has a global $U(1)$ symmetry

$$q \to e^{i\alpha/3}\, q$$

acting on the Dirac spinor. This is baryon number.

There is another approximate symmetry called $U(1)_A$, with A stands for "axial."

Writing the Lagrangian in terms of L & R fields

$$\mathcal{L}^{QCD}_{n_f=1} = \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R - m_q(\bar{q}_L q_R + \bar{q}_R q_L)$$

If $m_q \to 0$, then

$$q_L \to e^{i\beta} q_L$$

$$q_R \to e^{-i\beta} q_R$$

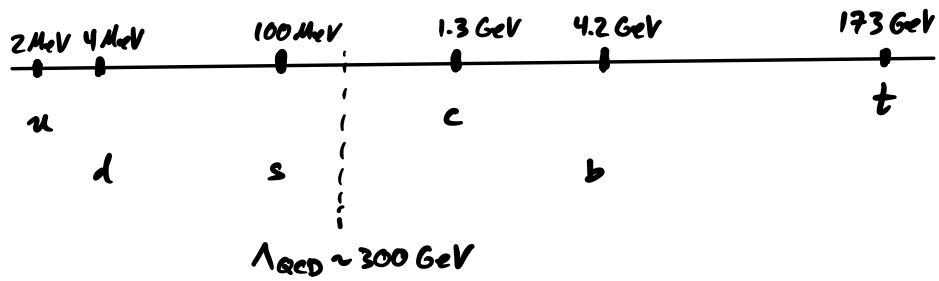
would be a symmetry. So the $m_q = 0$ the theory has a global symmetry

$$U(1)_L \times U(1)_R \quad w/ \quad \begin{aligned} q_L &\to e^{i\alpha_L} q_L \\ q_R &\to e^{i\alpha_R} q_R \end{aligned}$$

So $U(1)_B$ is the vector subgroup, $\alpha_L = \alpha_R$, while $U(1)_A$ is for $\alpha_L = -\alpha_R$

The $U(1)_L \times U(1)_R$ is called "chiral symmetry", and it is of use only if the parameter breaking it is small.

If we look at the quark masses,



it seems that, compared with the confinement scale Λ_{QCD}, up & down quark masses are much smaller, so their associated chiral symmetry is a good approximation. Also for the s-quark chiral symm is worth considering.

- $n_f = 2$

The first relevant case is when $n_f = 2$, so keeping only u & d. The doublet

$$q_\alpha = \begin{pmatrix} u \\ d \end{pmatrix}$$

can be used to write the $\mathcal{L}$ as

$$\mathcal{L}^{QCD}_{n_f=2} = -\frac{1}{2}\mathrm{tr}(G_{\mu\nu}G^{\mu\nu}) + \bar{q}^\alpha i\not{D} q_\alpha - \bar{q}^\alpha (m_q)_\alpha{}^\beta q_\beta$$

Note that the doublet has <u>nothing</u> to do with the EW doublet.

This theory, in the $m_q \to 0$ limit, has a larger global symmetry

$$q_{L,\alpha} \to (\Omega_L)_\alpha{}^\beta q_{L,\beta}$$
$$; \quad \Omega_L^\dagger \Omega_L = \Omega_R^\dagger \Omega_R = 1$$
$$q_{R,\alpha} \to (\Omega_R)_\alpha{}^\beta q_{R,\beta}$$

which is the chiral $U(2)_L \times U(2)_R$.

We can isolate an $SU(2)$ subgroup, the isospin group,

$$\Omega_L = \Omega_R = \Omega_I = e^{i\alpha_a \tau^a} \in SU(2)_I \subset U(2)_L \times U(2)_R$$

so it is the vector subgroup.

The chiral symmetry is broken by the quark masses.

But the isospin symm. is also broken by the mass difference between u & d quarks,

$$\left.\begin{array}{l} m_u \sim 2\,\text{MeV} \\ m_d \sim 4\,\text{MeV} \end{array}\right\} \quad m_u - m_d \sim 2\,\text{MeV}$$

Moreover, the isospin is broken by EW interactions, and in particular by QED. This is in fact the leading source of isospin breaking.

- $n_f = 3$

One can also take into account the s

quark. while

$$m_s \sim 100\ \text{MeV}$$

one still has

$$m_s \lesssim \Lambda_{QCD}$$

The quarks are now in a triplet

$$q_\alpha = \begin{pmatrix} u_\alpha \\ d_\alpha \\ s_\alpha \end{pmatrix}$$

with the mass matrix

$$m_q = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

& the chiral symm is

$$U(3)_L \times U(3)_R$$

Isospin is the vector $SU(2)$ acting on u & d.

While EW still break the chiral symm,

now the breaking from m_s dominates.

This breakin preserves isospin.

- So the $n_f = 3$ case has a $U(3)_L \times U(3)_R$ chiral symmetry, or

$$U(1)_B \times U(1)_A \times SU(3)_L \times SU(3)_R$$

global symmetry.

We know what $U(1)_B$ does to asymptotic states: separates mesons $(q\bar{q})$ from baryons (qqq).

- There is strong evidence that chiral symmetry is spontaneously broken by strong interactions. The vaccum contains an indefinite number of $q\bar{q}$ pairs. To preserve Lorentz, these pairs must form a scalar, so one has

$$\langle \bar{q}_{L,i}\, q_{R,j} \rangle \simeq \Lambda_{QCD}^3\, \delta_{ij}$$

There is no proof that this condensate forms, but as we shall see the consequences of it are in agreement with data.

The above vev means that chiral symm is spontaneously broken to the diagonal subgroup,

$$SU(3)_L \times SU(3)_R \longrightarrow SU(3)_V$$

The goldstone theorem predicts the existence of 8 massless goldstone bosons. Since the symm is approximate, the mass of Goldstones is not exactly zero.
Such light octet of pseudo-goldstone bosons exists & it is identified with the octet of light mesons.

- Current algebra

We shall explore some of the consequences of identifying the light mesons with the (pseudo-) Goldstone bosons of chiral symmetry. First we do it from the current algebra.

There are eight spont. broken axial charges

$$Q^{5a}, \quad a = 1, \dots, 8$$

with the corresponding axial currents A^a_μ.

The goldstones π^a are interpolated by these currents as

$$\langle 0 | A^a_\mu | \pi^b(p) \rangle = i f^{ab} p_\mu$$

Assuming $SU(3)_V$ symmetry unbroken, we can write $f^{ab} = f \delta^{ab}$, with f being the scale asociated to the goldstone.

Taking the divergence,

$$\langle 0 | \partial^\mu A^a_\mu(0) | \pi^b(p) \rangle = \delta^{ab} f m_\pi^2$$

so $m_\pi^2 = 0$ if the symmetry is not explicitly broken. However, if $m_\pi^2 \neq 0$, we can rewrite this as

$$\langle 0|\partial^\mu A_\mu^a(0)|\pi^b(p)\rangle = f\, m_\pi^2 \langle 0|\phi^a(0)|\pi^b(p)\rangle$$

Generalization of this to an operator relation is the partially-conserved axial-vector current (PCAC) hypothesis:

$$\partial^\mu A_\mu^a = f\, m_\pi^2\, \phi^a$$

The relation between the divergence of the current & pion fields leads to many relations.

For instance, using the equation above & writing the state $|\pi^b(p)\rangle$ using LSZ, one gets

$$\begin{aligned}
\delta_{ab}\, m_a^2 f_a &= \langle 0|\partial^\mu A_\mu^a(0)|\pi^b(p)\rangle \\
&= \frac{i(m_b^2-k^2)}{f_b m_b^2}\int d^4x\, e^{-ik\cdot x}\langle 0|T(\partial^\mu A_\mu^a(0)\,\partial^\nu A_\nu^b(x))|0\rangle \\
&= \frac{i(m_b^2\cdot k^2)}{f_b m_b^2}\Big(ik_\nu \int d^4x\, \langle 0|T\,\partial^\mu A_\mu^a(0)\, A_\nu^b(x)|0\rangle \\
&\quad - \int d^4x\, e^{-ik\cdot x}\langle 0|\delta(x_0)[A_0^b(x),\, \partial^\mu A_\mu^a(0)]|0\rangle\Big)
\end{aligned}$$

where we used

$$T(\partial^\nu A_\nu^a(x)\,\mathcal{O}(0)) = \partial^\nu T(A_\nu^a(x)\,\mathcal{O}(0))$$
$$- \delta(x^0)\,[A_\nu^a(x), \mathcal{O}(0)]$$

with the second term coming from taking ∂'s of the theta fn.

Now we take the low energy limit of this relation, $k \to 0$. We have

$$\delta_{ab}\, m_a^2 f_a^2 = i \int d^3x \, \langle 0 | [A_0^b(0,\vec{x}), \partial^\mu A_\mu^a(0)] | 0 \rangle$$

$$= i \langle 0 | [Q^{5b}, \partial^\mu A_\mu^a(0)] | 0 \rangle$$

The part with the spatial part of the comm is $[A_0^a, \partial^i A_i^b] \sim f^{abc} V_i^c$, that vanishes due to δ^{ab}.
The rest can be written as

$$\partial^0 A_0^a(0) = i [H, A_0^a(0)]$$ Ham. density

$$= i \int d^3y \, [\mathcal{H}(0,\vec{y}), A_0^a(0)]$$

$$= i \int d^3y \, [\mathcal{H}(0), A_0^a(0,\vec{y})]$$

$$= i \, [\mathcal{H}(0), Q^{5a}]$$

we finally get

$$\delta_{ab} m_a^2 f_a^2 = \langle 0 | [Q^{5a}, [Q^{5b}, \mathcal{H}]] | 0 \rangle$$

So the masses are given by a double commutator of the axial charge with the Hamiltonian density.

The part of $\mathcal{H}$ that breaks the symmetry is given by the quark mass terms,

$$\mathcal{H} = m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s$$

$$= c^0 u^0 + c^3 u^3 + c^8 u^8$$

The u's are scalar densities

$$u^i = \bar{q} \lambda_i q$$

with λ^i's being Gell-Mann matrices and $\lambda^0 = \sqrt{2/3}\, \mathbb{1}$. We also define

$$v^i = -i \bar{q} \lambda_i \gamma^5 q$$

The double commutator can be computed using

$$[Q^{5a}, u^j] = -i d_{ajk} v_k$$

$$[Q^{5a}, v^j] = -i d_{ajk} u_k$$

with the symmetric symbol d_{abc} being

$$\{T^a, T^b\} = \frac{1}{3}\delta^{ab}\mathbb{1} + d^{abc}T^c$$

and $d_{0ab} = \delta^{ab}\sqrt{2/3}$.

In the quark model,

$$u^0 = \sqrt{\frac{2}{3}}(\bar{u}u + \bar{d}d + \bar{s}s)$$

$$u^3 = \bar{u}u - \bar{d}d$$

$$u^8 = \sqrt{\frac{1}{3}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$$

with the coefficients being the quark masses

$$c_0 = \frac{1}{\sqrt{6}}(m_u + m_d + m_s)$$

$$c_3 = \frac{1}{3}(m_u - m_d)$$

$$c_8 = \frac{1}{\sqrt{3}}\left(\frac{m_u + m_d}{2} - m_s\right)$$

in the isospin limit, $m_u = m_d$ & $c_3 = 0$.

The breaking of chiral symm by m_s leads to three different masses:

$\pi : a = 1, 2, 3 \qquad K : a = 4, 5, 6, 7 \qquad \eta : a = 8$

Then,

$$f_\pi^2 m_\pi^2 = \frac{m_u + m_d}{2} \langle 0 | \bar{u}u + \bar{d}d | 0 \rangle$$

$$f_K^2 m_K^2 = \frac{m_u + m_s}{2} \langle 0 | \bar{u}u + \bar{s}s | 0 \rangle$$

$$f_\eta^2 m_\eta^2 = \frac{m_u + m_d}{6} \langle 0 | \bar{u}u + \bar{d}d | 0 \rangle + \frac{4 m_s}{3} \langle 0 | \bar{s}s | 0 \rangle$$

Assuming the vacuum to be SU(3) symmetric,

$$\langle 0 | \bar{u}u | 0 \rangle = \langle 0 | \bar{d}d | 0 \rangle = \langle 0 | \bar{s}s | 0 \rangle \equiv \mu^3$$

we get $f_\pi = f_K = f_\eta \equiv f$, and we recover the Gell-Mann - Okubo mass relation

$$4 m_K^2 = 3 m_\eta^2 + m_\pi^2$$

Moreover, we obtain the ratio of quark masses

$$\frac{m_u + m_d}{2 m_s} = \frac{m_\pi^2}{2 m_K^2 - m_\pi^2} \approx \frac{1}{25}$$

So $m_s \gg m_u, m_d$.

In terms of the parameters in the Hamiltonian,

$$\frac{c_8}{c_0} \approx -1.25 \qquad (SU(2)_L \times SU(2)_R \text{ symm}: -\sqrt{2})$$

Adding isospin breaking can be done by adding a $\mathcal{H}_\gamma$ contribution from QED.

One gets

$$f^2 m_{\pi^\pm}^2 = (m_u + m_d)\mu^3 + \mu_\gamma^3$$
$$f^2 m_{\pi^0}^2 = (m_u + m_d)\mu^3$$
$$f^2 m_{K^\pm}^2 = (m_u + m_s)\mu^3 + \mu_\gamma^3$$
$$f^2 m_{K^0}^2 = (m_d + m_s)\mu^3$$
$$f^2 m_\eta^2 = \tfrac{1}{3}(m_u + m_d + 4 m_s)\mu^3$$

leads to

$$\frac{m_d}{m_u} = \frac{m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - 2m_{\pi^0}^2} \simeq 1.8$$

So the u & d quark masses are very different, but the isospin breaking is small. The only solution is that $m_u, m_d \ll \Lambda_{QCD}$

- The chiral Lagrangian

There is a different, perhaps more direct way to understand the relations above.

The idea is to use what we learned from SSB. We saw that we can have a model with Σ,

$$\Sigma \to L \Sigma R^{\dagger}$$

with $L \in SU(3)_L$, $R \in SU(3)_R$, and give a potential to Σ to get SSB. We ended up with a $\mathcal{L}$ that, using the polar representation

$$\Sigma = (v+\sigma)\,\mathcal{U}, \quad \mathcal{U} = e^{i\frac{\tau\cdot\pi}{v}}$$

the goldstones were decoupled from the potential.

In our case, we have the condensate

$$\langle \bar{q}_L q_R \rangle = \Lambda_{QCD}^3 \, U_{ij}$$

with $U \in SU(3)$, a manifold of vacua.

So we have the matrix

$$U(x) = e^{\frac{i T^a \pi^a(x)}{f_\pi}}$$

parametrizing the Goldstones, transf. as

$$U \to L^\dagger U R$$

We can try to write the most general Lagrangian. It turns out that, at dim 4, there is a single operator,

$$\mathcal{L}_2 = \frac{f^2}{4} tr(\partial^\mu U^\dagger \partial_\mu U)$$

This includes the kinetic term for the goldstones, but also derivative interactions.

The matrix of Goldstones is given by

$$\pi^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

The chiral symm is broken by quark masses. The mass matrix can be thought of a parameter that transforms as

$$M \to L M R^\dagger$$

but it breaks the symm by taking

a specific value. Then, a term

$$\mathcal{L}_M = \frac{\sigma}{2} \, tr(MU + M^\dagger U^\dagger)$$

can be written. This gives a mass to the pion proportional to

$$m_\pi \sim \frac{\sigma}{f^2} \, tr(M) \, \pi^2$$

Explicitly, writing

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix},$$

one gets

$$\begin{aligned} \mathcal{L}_M = -\frac{\sigma}{f^2} \Big[\; & \frac{m_u + m_d}{2} \left((\pi^0)^2 + \pi^+\pi^- \right) \\ & + (m_u + m_s) \, K^+ K^- \\ & + (m_d + m_s) \, \bar{K}^0 K^0 \\ & + \frac{1}{2} \, \frac{m_u + m_d + 4 m_s}{3} \, \eta^2 \\ & + \frac{1}{\sqrt{3}} (m_u - m_d) \, \pi^0 \eta \; \Big] \end{aligned}$$

This is equivalent to the relations using current algebra.

First, quark & meson masses cannot be directly related due to the unknown σ/f^2.

- However, meson masses scale linearly with quark masses.

- We get $$\frac{m_{K^+}^2 - m_{K^0}^2}{m_\pi^2} = \frac{m_u - m_d}{m_u + m_d}$$

and, for $m_u \simeq m_d$, the Gell-Mann-Okubo relation
$$4 m_{K^+}^2 \simeq 3 m_\eta^2 + m_\pi^2$$
$$\Downarrow$$

experiment: $$m_K \simeq 495 \text{ MeV}$$
$$\frac{1}{2}\sqrt{3m_\eta^2 + m_\pi^2} \simeq 480 \text{ MeV}$$

- $U(1)_A$

So we learned what happens with the $U(1)_B$ and $SU(3)_L \times SU(3)_R$ of the glob $U(3)_L \times U(3)_R$. What about $U(1)_A$? Why its associated would-be goldstone, the η', has twice the mass of η even if it has similar quark content? The η',

$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

is a singlet under $SU(3)_V$.

It turns out that $U(1)_A$ is not a symmetry, and η' is not massless in the chiral limit. We will not be able to explore this further.